

L^p solutions of BSDEs with a new kind of non-Lipschitz coefficients[☆]

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Abstract

In this paper, we are interested in solving multidimensional backward stochastic differential equations (BSDEs) with a new kind of non-Lipschitz coefficients. We establish an existence and uniqueness result of solutions in L^p ($p > 1$), which includes some known results as its particular cases.

Keywords: Backward stochastic differential equation, Non-Lipschitz coefficients, Mao's condition, Constantin's condition, L^p solution

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1. Introduction

In this paper, we consider the following multidimensional backward stochastic differential equation (BSDE for short in the remaining):

$$y_t = \xi + \int_t^T g(s, y_s, z_s) ds - \int_t^T z_s dB_s, \quad t \in [0, T], \quad (1)$$

where $T \geq 0$ is a constant called the time horizon, ξ is a k -dimensional random vector called the terminal condition, the random function $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbf{R}^k \times \mathbf{R}^{k \times d} \rightarrow \mathbf{R}^k$ is progressively measurable for each (y, z) , called the generator of BSDE (1), and B is a d -dimensional Brownian motion. The solution (y, z) is a pair of adapted processes. The triple (ξ, T, g) is called the coefficients (parameters) of BSDE (1).

Such equations, in the nonlinear case, were firstly introduced by [5], who established an existence and uniqueness result for solutions in L^2 to BSDEs under the Lipschitz assumption of the generator g . Since then, BSDEs have been studied with great interest, and they have gradually become an import mathematical tool in many fields such as financial mathematics, stochastic games and optimal control, etc. In particular, many efforts have been done in relaxing the Lipschitz hypothesis on g , for instance, [3] proved the existence of a solution in L^2 for (1) when $k = 1$ and g is only continuous and of linear growth in y and z , [4] obtained an existence and uniqueness result of

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a solution in L^2 for (1) where g satisfies some kind of non-lipschitz conditions, and [6] established an existence and uniqueness result of a solution in L^2 for (1) where g satisfies some kind of monotonicity conditions in y . Furthermore, [1] investigated the existence and uniqueness of a solution in L^p ($p > 1$) for (1) where the generator g satisfies the monotonicity condition put forward in [6].

This paper is interested in solving multidimensional BSDEs with a new kind of non-Lipschitz coefficients. We establish an existence and uniqueness result of solutions in L^p ($p > 1$) for BSDE (1) (see Theorem 1 in Section 3), which includes the corresponding results in [4],[2] and [5] as its particular cases. This paper is organized as follows. We introduce some preliminaries and lemmas in Section 2 and put forward and prove our main result in Section 3. Finally, Section 4 is devoted to the analysis of the new kind of non-Lipschitz coefficients, and some corollaries, remarks and examples are also given in this section.

2. Preliminaries and Lemmas

Let us first introduce some notations. First of all, let us fix two real numbers $T \geq 0$ and $p > 1$, and two positive integers k and d . Let (Ω, \mathcal{F}, P) be a probability space carrying a standard d -dimensional Brownian motion $(B_t)_{t \geq 0}$. Let $(\mathcal{F}_t)_{t \geq 0}$ be the natural σ -algebra generated by $(B_t)_{t \geq 0}$ and $\mathcal{F} = \mathcal{F}_T$. In this paper, the Euclidean norm of a vector $y \in \mathbf{R}^k$ will be defined by $|y|$, and for an $k \times d$ matrix z , we define $|z| = \sqrt{\text{Tr} z z^*}$, where z^* is the transpose of z . Let $\langle x, y \rangle$ represent the inner product of $x, y \in \mathbf{R}^k$. We denote by $L^p(\mathbf{R}^k)$ the set of all \mathbf{R}^k -valued and \mathcal{F}_T -measurable random vectors ξ such that $\mathbf{E}[|\xi|^p] < +\infty$, let $\mathcal{S}^p(0, T; \mathbf{R}^k)$ denote the set of \mathbf{R}^k -valued, adapted and continuous processes $(Y_t)_{t \in [0, T]}$ such that

$$\|Y\|_{\mathcal{S}^p} := \left(\mathbf{E} \left[\sup_{t \in [0, T]} |Y_t|^p \right] \right)^{1/p} < +\infty.$$

Moreover, let $M^p(0, T; \mathbf{R}^k)$ (resp. $M^p(0, T; \mathbf{R}^{k \times d})$) denote the set of (equivalent classes of) (\mathcal{F}_t) -progressively measurable \mathbf{R}^k -valued ($\mathbf{R}^{k \times d}$ -valued) processes $(Z_t)_{t \in [0, T]}$ such that

$$\|Z\|_{M^p} := \left\{ \mathbf{E} \left[\left(\int_0^T |Z_t|^2 dt \right)^{p/2} \right] \right\}^{1/p} < +\infty.$$

Obviously, both \mathcal{S}^p and M^p are Banach spaces. As mentioned in the introduction, we will deal only with BSDEs which are equations of type (1), where the terminal condition ξ belongs to the space $L^p(\mathbf{R}^k)$, and the generator g is (\mathcal{F}_t) -progressively measurable for each (y, z) .

Definition 1 A pair of processes $(y_t, z_t)_{t \in [0, T]}$ is called a solution in L^p to BSDE (1), if $(y_t, z_t)_{t \in [0, T]} \in \mathcal{S}^p(0, T; \mathbf{R}^k) \times M^p(0, T; \mathbf{R}^{k \times d})$ and satisfies (1).

The following Lemma 1 comes from Corollary 2.3 in [1], which is the starting point of this paper.

Lemma 1 If $(y_t, z_t)_{t \in [0, T]}$ be a solution in L^p of BSDE (1), $c(p) = p[(p-1) \wedge 1]/2$ and $0 \leq t \leq T$, then

$$\begin{aligned} |y_t|^p + c(p) \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2 \, ds &\leq |\xi|^p + p \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} \langle y_s, g(s, y_s, z_s) \rangle \, ds \\ &\quad - p \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} \langle y_s, z_s dB_s \rangle. \end{aligned}$$

Now, let us introduce the following Proposition 1 and Proposition 2, which will play an important role in the proof of our main result. Before that, let us first introduce the following assumption on the generator g :

(A) $dP \times dt - a.e., \forall (y, z) \in \mathbf{R}^k \times \mathbf{R}^{k \times d}, |g(\omega, t, y, z)| \leq \psi^{\frac{1}{p}}(|y|^p) + \lambda|z| + \varphi_t + f_t$,

where $\lambda \geq 0$, both φ_t and f_t are two nonnegative, (\mathcal{F}_t) -progressively measurable processes with $\mathbf{E} \left[\int_0^T \varphi_t^p \, dt \right] < +\infty$ and $\mathbf{E} \left[\left(\int_0^T f_t \, dt \right)^p \right] < +\infty$, and $\psi(\cdot) : \mathbf{R}^+ \mapsto \mathbf{R}^+$ is a nondecreasing and concave function with $\psi(0) = 0$.

Proposition 1 Let assumption (A) hold and let $(y_t, z_t)_{t \in [0, T]}$ be a solution in L^p to BSDE (1). Then there exists a constant $C_{\lambda, p, T}$ depending on λ, p and T such that for each $t \in [0, T]$,

$$\begin{aligned} \mathbf{E} \left[\left(\int_t^T |z_s|^2 \, ds \right)^{p/2} \right] &\leq C_{\lambda, p, T} \left\{ \mathbf{E} \left[\sup_{s \in [t, T]} |y_s|^p \right] + \psi \left(\mathbf{E} \left[\sup_{s \in [t, T]} |y_s|^p \right] \right) \right. \\ &\quad \left. + \mathbf{E} \left[\int_t^T \varphi_s^p \, ds \right] + \mathbf{E} \left[\left(\int_t^T f_s \, ds \right)^p \right] \right\}. \end{aligned}$$

Proof. Applying Itô's formula to $|y_t|^2$ leads to that

$$|y_t|^2 + \int_t^T |z_s|^2 \, ds = |\xi|^2 + 2 \int_t^T \langle y_s, g(s, y_s, z_s) \rangle \, ds - 2 \int_t^T \langle y_s, z_s dB_s \rangle.$$

By assumption (A) we have, for each $s \in [t, T]$,

$$\begin{aligned} 2 \langle y_s, g(s, y_s, z_s) \rangle &\leq 2|y_s| \left(\psi^{\frac{1}{p}}(|y_s|^p) + \lambda|z_s| + \varphi_s + f_s \right) \\ &\leq 2 \left(\sup_{s \in [t, T]} |y_s| \right) \left(\psi^{\frac{1}{p}}(|y_s|^p) + \varphi_s + f_s \right) + 2\lambda^2 \sup_{s \in [t, T]} |y_s|^2 + \frac{|z_s|^2}{2}. \end{aligned}$$

Thus, in view of the inequality that $2ab \leq a^2 + b^2$ we can get that

$$\begin{aligned} \frac{1}{2} \int_t^T |z_s|^2 \, ds &\leq (3 + 2\lambda^2 T) \sup_{s \in [t, T]} |y_s|^2 + \left[\int_t^T \psi^{\frac{1}{p}}(|y_s|^p) \, ds \right]^2 \\ &\quad + \left[\int_t^T (\varphi_s + f_s) \, ds \right]^2 + 2 \left| \int_t^T \langle y_s, z_s dB_s \rangle \right|. \end{aligned}$$

Then noticing that $\psi(\cdot)$ is a nondecreasing function, by the inequality $(a + b)^{p/2} \leq 2^p(a^{p/2} + b^{p/2})$ we have

$$\left[\int_t^T |z_s|^2 \, ds \right]^{p/2} \leq c_{\lambda,p,T} \left[\sup_{s \in [t,T]} |y_s|^p + \psi \left(\sup_{s \in [t,T]} |y_s|^p \right) + \left[\int_t^T (\varphi_s + f_s) \, ds \right]^p + \left| \int_t^T \langle y_s, z_s dB_s \rangle \right|^{p/2} \right], \quad (2)$$

where $c_{\lambda,p,T} = 2^{p+4}(3 + 2\lambda^2 T + T^p)$ and we have used the fact that

$$\left[\int_t^T \psi^{\frac{1}{p}}(|y_s|^p) \, ds \right]^p \leq T^p \psi \left(\sup_{s \in [t,T]} |y_s|^p \right).$$

But by the Burkholder-Davis-Gundy (BDG) inequality, we get that for each $t \in [0, T]$,

$$\begin{aligned} c_{\lambda,p,T} \mathbf{E} \left[\left| \int_t^T \langle y_s, z_s dB_s \rangle \right|^{p/2} \right] &\leq d_{\lambda,p,T} \mathbf{E} \left[\left(\int_t^T |y_s|^2 |z_s|^2 \, ds \right)^{p/4} \right] \\ &\leq d_{\lambda,p,T} \mathbf{E} \left[\sup_{s \in [t,T]} |y_s|^{p/2} \cdot \left(\int_t^T |z_s|^2 \, ds \right)^{p/4} \right] \end{aligned}$$

and thus,

$$c_{\lambda,p,T} \mathbf{E} \left[\left| \int_t^T \langle y_s, z_s dB_s \rangle \right|^{p/2} \right] \leq \frac{d_{\lambda,p,T}^2}{2} \mathbf{E} \left[\sup_{s \in [t,T]} |y_s|^p \right] + \frac{1}{2} \mathbf{E} \left[\left(\int_t^T |z_s|^2 \, ds \right)^{p/2} \right].$$

Coming back to estimate (2) we get, for each $t \in [0, T]$,

$$\begin{aligned} \mathbf{E} \left[\left(\int_t^T |z_s|^2 \, ds \right)^{p/2} \right] &\leq \bar{C}_{\lambda,p,T} \left\{ \mathbf{E} \left[\sup_{s \in [t,T]} |y_s|^p \right] + \mathbf{E} \left[\psi \left(\sup_{s \in [t,T]} |y_s|^p \right) \right] \right. \\ &\quad \left. + \mathbf{E} \left[\left(\int_t^T (\varphi_s + f_s) \, ds \right)^p \right] \right\}. \end{aligned}$$

Thus noticing that $\psi(\cdot)$ is a concave function, we can deduce the desired conclusion from Jensen's inequality and Hölder's inequality. The proof is complete. \square

Proposition 2 Let assumption (A) hold and let $(y_t, z_t)_{t \in [0,T]}$ be a solution in L^p to BSDE (1). Then there exists constants $m_p > 0$ (depending on p) and $K_{\lambda,p} > 0$ (depending on λ and p) such that for each $t \in [0, T]$,

$$\begin{aligned} \mathbf{E} \left[\sup_{s \in [t,T]} |y_s|^p \right] &\leq e^{K_{\lambda,p}(T-t)} \left\{ m_p \mathbf{E}[|\xi|^p] + m_p \mathbf{E} \left[\left(\int_t^T f_s \, ds \right)^p \right] \right. \\ &\quad \left. + \frac{1}{2} \mathbf{E} \left[\int_t^T \varphi_s^p \, ds \right] + \frac{1}{2} \int_t^T \psi(\mathbf{E}[|y_s|^p]) \, ds \right\}. \end{aligned}$$

Proof. From Lemma 1, we get the following inequality:

$$\begin{aligned} |y_t|^p + c(p) \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2 \, ds &\leq |\xi|^p + p \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} \langle y_s, g(s, y_s, z_s) \rangle \, ds \\ &\quad - p \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} \langle y_s, z_s dB_s \rangle. \end{aligned}$$

Assumption (A) yields the inequality

$$\langle y_s, g(s, y_s, z_s) \rangle \leq |y_s| [\psi^{\frac{1}{p}}(|y_s|^p) + \lambda |z_s| + \varphi_s + f_s],$$

from which we deduce that, with probability one, for each $t \in [0, T]$,

$$\begin{aligned} |y_t|^p + c(p) \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2 \, ds &\leq |\xi|^p - p \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} \langle y_s, z_s dB_s \rangle \\ &\quad + p \int_t^T |y_s|^{p-1} [\psi^{\frac{1}{p}}(|y_s|^p) + \lambda |z_s| + \varphi_s + f_s] \, ds. \end{aligned}$$

First of all, in view of the fact that $\psi(\cdot)$ increases at most linearly since it is a nondecreasing concave function and $\psi(0) = 0$, we deduce from the previous inequality that, $dP - a.s.$,

$$\int_0^T |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2 \, ds < +\infty.$$

Moreover, making use of Young's inequality ($a^r b^{1-r} \leq ra + (1-r)b$ for each $a \geq 0$, $b \geq 0$ and $r \in (0, 1)$) and the inequality $(a+b)^p \leq 2^p(a^p + b^p)$ we can obtain

$$\begin{aligned} p \int_t^T |y_s|^{p-1} (\psi^{\frac{1}{p}}(|y_s|^p) + \varphi_s) \, ds &= p \int_t^T \left\{ \left(\theta^{\frac{1}{p-1}} |y_s|^p \right)^{\frac{p-1}{p}} \cdot \left[\frac{1}{\theta} \left(\psi^{\frac{1}{p}}(|y_s|^p) + \varphi_s \right)^p \right]^{\frac{1}{p}} \right\} \, ds \\ &\leq (p-1) \theta^{\frac{1}{p-1}} \int_t^T |y_s|^p \, ds + \frac{2^p}{\theta} \int_t^T (\psi(|y_s|^p) + \varphi_s^p) \, ds, \end{aligned}$$

where $\theta > 0$ will be chosen later. And, from the inequality that $ab \leq (a^2 + b^2)/2$ we get that

$$\begin{aligned} p\lambda |y_s|^{p-1} |z_s| &= p \left(\frac{\sqrt{2}\lambda}{\sqrt{1 \wedge (p-1)}} |y_s|^{\frac{p}{2}} \right) \left(\sqrt{\frac{1 \wedge (p-1)}{2}} |y_s|^{\frac{p-2}{2}} |z_s| \right) \\ &\leq \frac{p\lambda^2}{1 \wedge (p-1)} |y_s|^p + \frac{c(p)}{2} |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2. \end{aligned}$$

Thus for each $t \in [0, T]$, we have

$$|y_t|^p + \frac{c(p)}{2} \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2 \, ds \leq X_t - p \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} \langle y_s, z_s dB_s \rangle, \quad (3)$$

where $X_t = |\xi|^p + d_{\lambda,p,\theta} \int_t^T |y_s|^p \, ds + \frac{2^p}{\theta} \int_t^T (\psi(|y_s|^p) + \varphi_s^p) \, ds + p \int_t^T |y_s|^{p-1} f_s \, ds$ with $d_{\lambda,p,\theta} = (p-1)\theta^{1/(p-1)} + p\lambda^2/[1 \wedge (p-1)] > 0$.

It follows from the BDG inequality that $\{M_t := \int_0^t |y_s|^{p-2} 1_{|y_s| \neq 0} \langle y_s, z_s dB_s \rangle\}_{t \in [0, T]}$ is a uniformly integrable martingale. In fact, we have, by Young's inequality,

$$\begin{aligned} \mathbf{E} \left[\langle M, M \rangle_T^{1/2} \right] &\leq \mathbf{E} \left[\sup_{s \in [0, T]} |y_s|^{p-1} \cdot \left(\int_0^T |z_s|^2 \, ds \right)^{1/2} \right] \\ &= \mathbf{E} \left\{ \left(\sup_{s \in [0, T]} |y_s|^p \right)^{\frac{p-1}{p}} \cdot \left[\left(\int_0^T |z_s|^2 \, ds \right)^{p/2} \right]^{\frac{1}{p}} \right\} \\ &\leq \frac{(p-1)}{p} \mathbf{E} \left[\sup_{s \in [0, T]} |y_s|^p \right] + \frac{1}{p} \mathbf{E} \left[\left(\int_0^T |z_s|^2 \, ds \right)^{p/2} \right] < +\infty. \end{aligned}$$

Coming back to inequality (3), and taking the expectation, we get both

$$\frac{c(p)}{2} \mathbf{E} \left[\int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2 \, ds \right] \leq \mathbf{E}[X_t] \quad (4)$$

and

$$\mathbf{E} \left[\sup_{s \in [t, T]} |y_s|^p \right] \leq \mathbf{E}[X_t] + k_p \mathbf{E} \left[(\langle M, M \rangle_T - \langle M, M \rangle_t)^{1/2} \right], \quad (5)$$

where we have used the BDG inequality in the last inequality.

On the other hand, making use of Young's inequality we have also

$$\begin{aligned} & k_p \mathbf{E} \left[(\langle M, M \rangle_T - \langle M, M \rangle_t)^{1/2} \right] \\ & \leq k_p \mathbf{E} \left[\sup_{s \in [t, T]} |y_s|^{p/2} \cdot \left(\int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2 \, ds \right)^{1/2} \right] \\ & \leq \frac{1}{2} \mathbf{E} \left[\sup_{s \in [t, T]} |y_s|^p \right] + \frac{k_p^2}{2} \mathbf{E} \left[\int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2 \, ds \right]. \end{aligned}$$

Coming back to inequalities (4) and (5), we obtain

$$\mathbf{E} \left[\sup_{s \in [t, T]} |y_s|^p \right] \leq k'_p \mathbf{E}[X_t].$$

Applying once again Young's inequality, we get

$$\begin{aligned} pk'_p \mathbf{E} \left[\int_t^T |y_s|^{p-1} f_s \, ds \right] & \leq pk'_p \mathbf{E} \left[\sup_{s \in [t, T]} |y_s|^{p-1} \int_t^T f_s \, ds \right] \\ & = \mathbf{E} \left\{ \left(\frac{p}{2(p-1)} \sup_{s \in [t, T]} |y_s|^p \right)^{\frac{p-1}{p}} \cdot \left[\frac{pk_p''}{2} \left(\int_t^T f_s \, ds \right)^p \right]^{\frac{1}{p}} \right\} \\ & \leq \frac{1}{2} \mathbf{E} \left[\sup_{s \in [t, T]} |y_s|^p \right] + \frac{k_p''}{2} \mathbf{E} \left[\left(\int_t^T f_s \, ds \right)^p \right], \end{aligned}$$

from which we deduce, coming back to the definition of X_t , that

$$\begin{aligned} \mathbf{E} \left[\sup_{s \in [t, T]} |y_s|^p \right] & \leq 2k'_p \mathbf{E} \left[|\xi|^p + d_{\lambda, p, \theta} \int_t^T |y_s|^p \, ds + \frac{2^p}{\theta} \int_t^T (\psi(|y_s|^p) + \varphi_s^p) \, ds \right] \\ & \quad + k_p'' \mathbf{E} \left[\left(\int_t^T f_s \, ds \right)^p \right]. \end{aligned}$$

By letting $\theta = 2^{p+2}k'_p$ and $h_t = \mathbf{E} \left[\sup_{s \in [t, T]} |y_s|^p \right]$ in the previous inequality and using Fubini theorem and Jensen's inequality, noticing that $\psi(\cdot)$ is a concave function, we have, for each $t \in [0, T]$,

$$\begin{aligned} h_t & \leq 2k'_p \mathbf{E}[|\xi|^p] + k_p'' \mathbf{E} \left[\left(\int_t^T f_s \, ds \right)^p \right] + \frac{1}{2} \mathbf{E} \left[\int_t^T \varphi_s^p \, ds \right] \\ & \quad + \frac{1}{2} \int_t^T \psi(\mathbf{E}[|y_s|^p]) \, ds + 2k'_p d_{\lambda, p, \theta} \int_t^T h_s \, ds. \end{aligned}$$

Finally, Gronwall's inequality yields that for each $t \in [0, T]$,

$$h_t \leq e^{2k'_p d_{\lambda,p,\theta}(T-t)} \left\{ 2k'_p \mathbf{E}[|\xi|^p] + k''_p \mathbf{E} \left[\left(\int_t^T f_s \, ds \right)^p \right] + \frac{1}{2} \mathbf{E} \left[\int_t^T \varphi_s^p \, ds \right] + \frac{1}{2} \int_t^T \psi(\mathbf{E}[|y_s|^p]) \, ds \right\}.$$

Then we complete the proof of Proposition 2. \square

3. Main Result and Its Proof

In this section, we will put forward and prove our main result. Let us first introduce the following assumptions:

(H1) There exists a nondecreasing and concave function $\rho(\cdot) : \mathbf{R}^+ \mapsto \mathbf{R}^+$ with $\rho(0) = 0$, $\rho(u) > 0$ for $u > 0$ and $\int_{0+} \frac{du}{\rho(u)} = +\infty$ such that $dP \times dt - a.e.$,

$$\forall y_1, y_2 \in \mathbf{R}^k, z \in \mathbf{R}^{k \times d}, \quad |g(\omega, t, y_1, z) - g(\omega, t, y_2, z)|^p \leq \rho(|y_1 - y_2|^p).$$

(H2) There exists a constant $C \geq 0$ such that $dP \times dt - a.e.$,

$$\forall y \in \mathbf{R}^k, z_1, z_2 \in \mathbf{R}^{k \times d}, \quad |g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq C|z_1 - z_2|.$$

$$(H3) \quad \mathbf{E} \left[\left(\int_0^T |g(t, 0, 0)| \, dt \right)^p \right] < +\infty.$$

Remark 1 Since $\rho(\cdot)$ is a nondecreasing and concave function with $\rho(0) = 0$, it increases at most linearly, i.e., there exists a constant $A > 0$ such that $\rho(x) \leq A(x+1)$ for each $x \geq 0$.

The following Theorem 1 is the main result of this paper.

Theorem 1 Let g satisfy assumptions (H1)-(H3). Then for each $\xi \in L^p(\mathbf{R}^k)$, the BSDE with parameters (ξ, T, g) has a unique solution in L^p .

We can construct the Picard approximate sequence of the BSDE with parameters (ξ, T, g) as follows:

$$y_t^0 = 0; \quad y_t^n = \xi + \int_t^T g(s, y_s^{n-1}, z_s^n) ds - \int_t^T z_s^n dB_s, \quad t \in [0, T]. \quad (6)$$

Indeed, for each $n \geq 1$, by (H1) and Remark 1 we have

$$\begin{aligned} |g(s, y_s^{n-1}, 0)| &\leq |g(s, 0, 0)| + \rho^{\frac{1}{p}}(|y_s^{n-1}|^p) \\ &\leq |g(s, 0, 0)| + A^{\frac{1}{p}}(|y_s^{n-1}| + 1), \end{aligned}$$

and then

$$\begin{aligned} \mathbf{E} \left[\left(\int_0^T |g(s, y_s^{n-1}, 0)| \, ds \right)^p \right] &\leq 2^p \mathbf{E} \left[\left(\int_0^T |g(s, 0, 0)| \, ds \right)^p \right] \\ &\quad + A(2T)^p \left(\mathbf{E} \left[\sup_{s \in [0, T]} |y_s^{n-1}|^p \right] + 1 \right). \end{aligned}$$

Then the generator $g(t, y_t^{n-1}, z)$ of BSDE (6) satisfies (H2) and (H3). It follows from Theorem 4.2 in [1] that, for each $n \geq 1$, equation (6) has a unique solution $(y_t^n, z_t^n)_{t \in [0, T]}$ in L^p . Concerning the processes $(y_t^n, z_t^n)_{t \in [0, T]}$, we have the following Lemma 2 and Lemma 3.

Lemma 2 Under the hypotheses of Theorem 1, there exists a constant $c_1 > 0$ depending only on C and p , and a constant $K > 0$ depending only on C, p and T , such that for each $t \in [0, T], n, m \geq 1$,

$$\mathbf{E} \left[\sup_{s \in [t, T]} |y_s^{n+m} - y_s^n|^p \right] \leq \frac{1}{2} e^{c_1(T-t)} \int_t^T \rho(\mathbf{E}[|y_s^{n+m-1} - y_s^{n-1}|^p]) \, ds. \quad (7)$$

and

$$\begin{aligned} \mathbf{E} \left[\left(\int_t^T |z_s^{n+m} - z_s^n|^2 \, ds \right)^{p/2} \right] &\leq K \left\{ \mathbf{E} \left[\sup_{s \in [t, T]} |y_s^{n+m} - y_s^n|^p \right] \right. \\ &\quad \left. + \int_t^T \rho(\mathbf{E}[|y_s^{n+m-1} - y_s^{n-1}|^p]) \, ds \right\}. \end{aligned} \quad (8)$$

Proof. It follows from (6) that the process $(y_t^{n+m} - y_t^n, z_t^{n+m} - z_t^n)_{t \in [0, T]}$ is a solution in L^p of the following BSDE

$$y_t = \int_t^T f_{n,m}(s, z_s) \, ds - \int_t^T z_s \, dB_s, \quad t \in [0, T] \quad (9)$$

where

$$f_{n,m}(s, z) := g(s, y_s^{n+m-1}, z + z_s^n) - g(s, y_s^{n-1}, z_s^n).$$

By (H1) and (H2) we have

$$|f_{n,m}(s, z)| \leq \rho^{\frac{1}{p}}(|y_s^{n+m-1} - y_s^{n-1}|^p) + C|z|,$$

which means that assumption (A) is satisfied for the generator $f_{n,m}(t, z)$ of BSDE (9) with $\psi(\cdot) \equiv 0$, $\lambda = C$, $f_t \equiv 0$ and $\varphi_t = \rho^{\frac{1}{p}}(|y_t^{n+m-1} - y_t^{n-1}|^p)$ by Remark 1. Thus, the conclusions (7) and (8) follows, in view of the fact that $\rho(\cdot)$ is a concave function, from Proposition 2 and Proposition 1, and then Fubini theorem and Jensen's inequality. \square

Lemma 3 Under the hypotheses of Theorem 1, there exists $T_1 \in [0, T]$ independent of the terminal condition ξ and a constant $M \geq 0$ such that for each $n \geq 1, t \in [T_1, T]$,

$$\mathbf{E} \left[\sup_{r \in [t, T]} |y_r^n|^p \right] \leq M.$$

Proof. Making use of the hypotheses of Theorem 1 we know that

$$\begin{aligned} |g(s, y_s^{n-1}, z)| &\leq |g(s, y_s^{n-1}, z) - g(s, 0, 0)| + |g(s, 0, 0)| \\ &\leq \rho^{\frac{1}{p}}(|y_s^{n-1}|^p) + C|z| + |g(s, 0, 0)| \end{aligned}$$

Thus, assumption (A) is satisfied for the generator $g(t, y_t^{n-1}, z)$ of BSDE (6) with $\psi(\cdot) \equiv 0$, $\lambda = C$, $f_t = |g(t, 0, 0)|$ by (H3) and $\varphi_t = \rho^{\frac{1}{p}}(|y_t^{n-1}|^p)$ by Remark 1. Consequently,

in view of the fact that $\rho(\cdot)$ is a concave function, from Proposition 2 and then Fubini theorem and Jensen's inequality we get that there exist two positive constants c_2 and c_3 depending only on C and p such that for each $n \geq 1$ and each $t \in [0, T]$,

$$\mathbf{E} \left[\sup_{r \in [t, T]} |y_r^n|^p \right] \leq \mu_t + \frac{1}{2} e^{c_3(T-t)} \int_t^T \rho(\mathbf{E}[|y_s^{n-1}|^p]) \, ds \quad (10)$$

where $\mu_t = c_2 e^{c_3(T-t)} \left\{ \mathbf{E}|\xi|^p + \mathbf{E} \left[\left(\int_t^T |g(s, 0, 0)| \, ds \right)^p \right] \right\} \geq 0$.

Now, let $M = 2\mu_0 + 2AT$ and $T_1 = \max\{T - \ln 2/c_1, T - \ln 2/c_3, T - 1/2A, 0\}$ where c_1 is defined in Lemma 2 and A is defined in Remark 1. Then for each $t \in [T_1, T]$, we have

$$\frac{1}{2} e^{c_1(T-t)} \leq 1, \quad \frac{1}{2} e^{c_3(T-t)} \leq 1 \quad \text{and} \quad A(T-t) \leq \frac{1}{2}. \quad (11)$$

Thus, from (10) and (11) we have

$$\mathbf{E} \left[\sup_{r \in [t, T]} |y_r^n|^p \right] \leq \mu_0 + \int_t^T \rho(\mathbf{E}[|y_s^{n-1}|^p]) \, ds, \quad t \in [T_1, T]. \quad (12)$$

Since $\rho(\cdot)$ is a nondecreasing function, by (12), Remark 1 and (11) we can deduce that, for each $t \in [T_1, T]$,

$$\mathbf{E} \left[\sup_{r \in [t, T]} |y_r^1|^p \right] \leq \mu_0 \leq M,$$

$$\mathbf{E} \left[\sup_{r \in [t, T]} |y_r^2|^p \right] \leq \mu_0 + \int_t^T \rho(M) \, ds \leq \mu_0 + A(M+1)(T-t) \leq \mu_0 + \frac{M}{2} + AT = M,$$

$$\mathbf{E} \left[\sup_{r \in [t, T]} |y_r^3|^p \right] \leq \mu_0 + \int_t^T \rho(M) \, ds \leq \mu_0 + A(M+1)(T-t) \leq \mu_0 + \frac{M}{2} + AT = M.$$

By induction, we can prove that for all $n \geq 1$ and all $t \in [T_1, T]$,

$$\mathbf{E} \left[\sup_{r \in [t, T]} |y_r^n|^p \right] \leq M.$$

The proof of Lemma 3 is complete. \square

With the help of Lemma 2 and Lemma 3, we can prove Theorem 1.

Proof of Theorem 1. Existence: Define a sequence of functions $\{\varphi_n(t)\}_{n \geq 1}$ as follows:

$$\varphi_0(t) = \int_t^T \rho(M) \, ds; \quad \varphi_{n+1}(t) = \int_t^T \rho(\varphi_n(s)) \, ds. \quad (13)$$

Then for all $t \in [T_1, T]$, from the proof of Lemma 3 we have

$$\begin{aligned} \varphi_0(t) &= \int_t^T \rho(M) \, ds \leq M, \\ \varphi_1(t) &= \int_t^T \rho(\varphi_0(s)) \, ds \leq \int_t^T \rho(M) \, ds = \varphi_0(t) \leq M, \\ \varphi_2(t) &= \int_t^T \rho(\varphi_1(s)) \, ds \leq \int_t^T \rho(\varphi_0(s)) \, ds = \varphi_1(t) \leq M. \end{aligned}$$

By induction, we can prove that for all $n \geq 1$, $\varphi_n(t)$ satisfies

$$0 \leq \varphi_{n+1}(t) \leq \varphi_n(t) \leq \cdots \leq \varphi_1(t) \leq \varphi_0(t) \leq M.$$

Then, for each $t \in [T_1, T]$, the limit of the sequence $\{\varphi_n(t)\}_{n \geq 1}$ must exist, we denote it by $\varphi(t)$. Thus, letting $n \rightarrow \infty$ in (13), in view of the facts that $\rho(\cdot)$ is a continuous function and $\rho(\varphi_n(s)) \leq \rho(M)$ for each $n \geq 1$, we can deduce from the Lebesgue dominated convergence theorem that for each $t \in [T_1, T]$,

$$\varphi(t) = \int_t^T \rho(\varphi(s)) \, ds.$$

Then Bihari's inequality (see Lemma 3.6 in [4]) yields that for each $t \in [T_1, T]$, $\varphi(t) = 0$.

Now, for all $t \in [T_1, T]$, $n, m \geq 1$, thanks to Lemma 3, (7) in Lemma 2 and inequality (11) we have,

$$\begin{aligned} \mathbf{E} \left[\sup_{r \in [t, T]} |y_r^n|^p \right] &\leq M, \\ \mathbf{E} \left[\sup_{r \in [t, T]} |y_r^{1+m} - y_r^1|^p \right] &\leq \int_t^T \rho(\mathbf{E}[|y_s^m|^p]) \, ds \leq \int_t^T \rho(M) \, ds = \varphi_0(t) \leq M, \\ \mathbf{E} \left[\sup_{r \in [t, T]} |y_r^{2+m} - y_r^2|^p \right] &\leq \int_t^T \rho(\mathbf{E}[|y_s^{1+m} - y_s^1|^p]) \, ds \leq \int_t^T \rho(\varphi_0(s)) \, ds = \varphi_1(t) \leq M, \\ \mathbf{E} \left[\sup_{r \in [t, T]} |y_r^{3+m} - y_r^3|^p \right] &\leq \int_t^T \rho(\mathbf{E}[|y_s^{2+m} - y_s^2|^p]) \, ds \leq \int_t^T \rho(\varphi_1(s)) \, ds = \varphi_2(t) \leq M. \end{aligned}$$

By induction, we can derive that

$$\mathbf{E} \left[\sup_{T_1 \leq r \leq T} |y_r^{n+m} - y_r^n|^p \right] \leq \varphi_{n-1}(T_1) \rightarrow 0, \quad n \rightarrow \infty.$$

which means that $\{y_t^n\}_{n \geq 1}$ is a Cauchy sequence in $\mathcal{S}^p(T_1, T; \mathbf{R}^k)$. Furthermore, in view of the fact that $\rho(\cdot)$ is a continuous function we also know from (8) in Lemma 2 that $\{z_t^n\}_{n \geq 1}$ is a Cauchy sequence in $M^p(T_1, T; \mathbf{R}^{k \times d})$. Define their limits by $(y_t)_{t \in [T_1, T]}$ and $(z_t)_{t \in [T_1, T]}$ respectively. Thus, by letting $n \rightarrow \infty$ in (6), we get that (y_t, z_t) is a solution in L^p to the BSDE with parameters (ξ, T, g) on $[T_1, T]$. Note from Lemma 3 that the T_1 does not depend on the terminal condition ξ . Hence we can deduce by iteration the existence on $[T - l(T - T_1), T]$, for each l , and therefore the existence on the whole $[0, T]$. The existence has been proved.

Uniqueness: Let $(y_t^i, z_t^i)_{t \in [0, T]}$ ($i = 1, 2$) be two solutions in L^p of the BSDE with parameters (ξ, T, g) . It follows that $(y_t^1 - y_t^2, z_t^1 - z_t^2)_{t \in [0, T]}$ is a solution in L^p to the following BSDE:

$$y_t = \int_t^T \hat{g}(s, y_s, z_s) ds - \int_t^T z_s dB_s, \quad t \in [0, T], \quad (14)$$

where

$$\hat{g}(s, y, z) := g(s, y + y_s^2, z + z_s^2) - g(s, y_s^2, z_s^2).$$

By (H1) and (H2) we have

$$|\hat{g}(s, y, z)| \leq \rho^{\frac{1}{p}}(|y|^p) + C|z|,$$

which means that assumption (A) is satisfied for the generator $\hat{g}(t, y, z)$ of BSDE (14) with $\psi(\cdot) = \rho(\cdot)$, $\lambda = C$, $\varphi_t \equiv 0$ and $f_t \equiv 0$. Then from Proposition 2 and Proposition 1 we can obtain that there exists a constant $c_4 > 0$ depending only on C and p , and a constant $c_5 > 0$ depending only on C , p and T , such that for $t \in [0, T]$,

$$\mathbf{E} [|y_t^1 - y_t^2|^p] \leq \frac{1}{2} e^{c_4(T-t)} \int_t^T \rho(\mathbf{E} [|y_s^1 - y_s^2|^p]) \, ds \quad (15)$$

and

$$\mathbf{E} \left[\left(\int_t^T |z_s^1 - z_s^2|^2 \, ds \right)^{p/2} \right] \leq c_5 \left\{ \mathbf{E} \left[\sup_{s \in [t, T]} |y_s^1 - y_s^2|^p \right] + \rho \left(\mathbf{E} \left[\sup_{s \in [t, T]} |y_s^1 - y_s^2|^p \right] \right) \right\}. \quad (16)$$

Then from (15) Bihari's inequality (see Lemma 3.6 in [4]) yields that for each $t \in [0, T]$, $\mathbf{E} [|y_t^1 - y_t^2|^p] = 0$. This means $y_t^1 = y_t^2$ for all $t \in [0, T]$ almost surely. We can immediately deduce that $z_t^1 = z_t^2$ for all $t \in [0, T]$ almost surely by (16). The proof of the Theorem 1 is then complete. \square

4. Corollaries, Remarks and Examples

In this section, we are devoted to the analysis of the new kind of non-Lipschitz coefficients. Some corollaries, remarks and examples are given to show that Theorem 1 of this paper is a generalization of the corresponding results in [4], [2] and [5]. Firstly, by Theorem 1 the following corollary is immediate. By Hölder's inequality we know that it generalizes Theorem 2.1 in [4] where (H3) is replaced with $g(\cdot, 0, 0) \in M^2(0, T; \mathbf{R}^k)$.

Corollary 1 Let g satisfy (H1) with $p = 2$, (H2) and (H3). Then for each $\xi \in L^2(\mathbf{R}^k)$, the BSDE with parameters (ξ, T, g) has a unique solution in L^2 .

Furthermore, by letting $\rho(x) = \mu^p x$ with $\mu > 0$ in (H1) we can obtain the following classical Lipschitz assumption in y with respect to the generator g :

(H1') There exists a constant $\mu \geq 0$ such that $dP \times dt - a.e.$,

$$\forall y_1, y_2 \in \mathbf{R}^k, z \in \mathbf{R}^{k \times d}, \quad |g(\omega, t, y_1, z) - g(\omega, t, y_2, z)| \leq \mu |y_1 - y_2|.$$

Consequently, by Theorem 1 we can get the following corollary, which generalizes the main result in [5] where $p = 2$.

Corollary 2 Let g satisfy (H1'), (H2) and (H3). Then for each $\xi \in L^p(\mathbf{R}^k)$, the BSDE with parameters (ξ, T, g) has a unique solution in L^p .

Remark 2 In the following, we will show that the concavity condition of $\rho(\cdot)$ in (H1) can be actually lifted and that the bigger the p , the stronger the (H1). To be precise, we need to prove that if g satisfy the following assumption (H1'') with $q \geq p$, then g must satisfy (H1).

(H1'') There exists a nondecreasing and continuous function $\kappa(\cdot) : \mathbf{R}^+ \mapsto \mathbf{R}^+$ with $\kappa(0) = 0$, $\kappa(u) > 0$ for $u > 0$ and $\int_{0+} \frac{du}{\kappa(u)} = +\infty$ such that $dP \times dt - a.e.$,

$$\forall y_1, y_2 \in \mathbf{R}^k, z \in \mathbf{R}^{k \times d}, \quad |g(\omega, t, y_1, z) - g(\omega, t, y_2, z)|^q \leq \kappa(|y_1 - y_2|^q).$$

In order to show Remark 2, we need the following technical Lemma. Its proof can be done by means of approximation procedures in [2], here we omit it.

Lemma 4 Let $\rho(\cdot)$ be a nondecreasing and concave function on \mathbf{R}^+ with $\rho(0) = 0$. Then we have

$$\forall r > 1, \quad \rho^r(x^{\frac{1}{r}}) \text{ is also a nondecreasing and concave function on } \mathbf{R}^+. \quad (17)$$

Moreover, if $\rho(u) > 0$ for $u > 0$ and $\int_{0+} \frac{du}{\rho(u)} = +\infty$, then

$$\forall r < 1, \quad \int_{0+} \frac{du}{\rho^r(u^{\frac{1}{r}})} = +\infty. \quad (18)$$

Now, we can show that (H1'') \implies (H1). Let us assume that (H1'') holds for g . Then we have, $dP \times dt - a.e.$,

$$\forall y_1, y_2 \in \mathbf{R}^k, z \in \mathbf{R}^{k \times d}, \quad |g(\omega, t, y_1, z) - g(\omega, t, y_2, z)| \leq \rho_1(|y_1 - y_2|),$$

where $\rho_1(u) := \kappa^{\frac{1}{q}}(u^q)$. Obviously, $\rho_1(u)$ is a continuous and nondecreasing function on \mathbf{R}_+ with $\rho_1(0) = 0$ and $\rho_1(x) > 0$ for $x > 0$, but it is not necessary to be concave. However, it follows from [2] that if g satisfies the above condition, then there exists a concave and nondecreasing function $\rho_2(\cdot)$ such that $\rho_2(0) = 0$, $\rho_2(u) \leq 2\rho_1(u)$ for $u \geq 0$, and $dP \times dt - a.e.$,

$$\forall y_1, y_2 \in \mathbf{R}^k, z \in \mathbf{R}^{k \times d}, \quad |g(\omega, t, y_1, z) - g(\omega, t, y_2, z)| \leq \rho_2(|y_1 - y_2|).$$

Thus, $dP \times dt - a.e.$,

$$\forall y_1, y_2 \in \mathbf{R}^k, z \in \mathbf{R}^{k \times d}, \quad |g(\omega, t, y_1, z) - g(\omega, t, y_2, z)|^p \leq \bar{\rho}(|y_1 - y_2|^p),$$

where $\bar{\rho}(u) := \rho_2^p(u^{\frac{1}{p}}) + u$. It is clear that $\bar{\rho}(0) = 0$ and $\bar{\rho}(u) > 0$ for $u > 0$. Moreover, it follows from (17) in Lemma 4 that $\bar{\rho}(\cdot)$ is also a nondecreasing and concave function since $p > 1$ and $\rho_2(\cdot)$ is a nondecreasing and concave function. Thus, to prove that (H1) holds, it suffices to show that $\int_{0+} \frac{du}{\bar{\rho}(u)} = +\infty$. Indeed, if $\rho_2(1) = 0$, then since $\rho_2(u) = 0$ for each $u \in [0, 1]$, we have

$$\int_{0+} \frac{du}{\bar{\rho}(u)} = \int_{0+} \frac{du}{u} = +\infty.$$

On the other hand, if $\rho_2(1) > 0$, since $\rho_2(\cdot)$ is a concave function with $\rho_2(0) = 0$, we know that $\forall u \in [0, 1]$, $\rho_2(u) \geq u\rho_2(1)$, and then

$$\rho_2^p(u^{\frac{1}{p}}) \geq \left(u^{\frac{1}{p}}\rho_2(1)\right)^p = \rho_2^p(1)u.$$

Thus, we have

$$\forall u \geq 0, \bar{\rho}(u) = \rho_2^p(u^{\frac{1}{p}}) + u \leq K\rho_2^p(u^{\frac{1}{p}}) \leq K2^p\rho_1^p(u^{\frac{1}{p}}) = K2^p\kappa^{\frac{p}{q}}(u^{\frac{q}{p}}), \quad (19)$$

where $K = 1 + 1/\rho_2^p(1)$. Consequently, if $q = p$, then

$$\int_{0+} \frac{du}{\bar{\rho}(u)} \geq \frac{1}{K2^p} \int_{0+} \frac{du}{\kappa(u)} = +\infty.$$

Thus, we have proved that (H1'') with $q = p$ implies (H1). Hence, now we can assume that the $\bar{\kappa}(\cdot)$ in (H1'') is a concave function. Then, if $q > p$, from (19) and (18) in Lemma 4 with $\rho(\cdot) = \kappa(\cdot)$ and $r = p/q < 1$ we have

$$\int_{0+} \frac{du}{\bar{\rho}(u)} \geq \frac{1}{K2^p} \int_{0+} \frac{du}{\kappa^{\frac{p}{q}}(u^{\frac{q}{p}})} = +\infty.$$

Thus, (H1) holds. Hence (H1'') \implies (H1), i.e., the concavity condition of $\rho(\cdot)$ in (H1) can be actually lifted and the bigger the p , the stronger the (H1).

Furthermore, let us introduce a stronger assumption (H1*) than (H1):

(H1*) There exists a nondecreasing and continuous function $\kappa(\cdot) : \mathbf{R}^+ \mapsto \mathbf{R}^+$ with $\kappa(0) = 0$, $\kappa(u) > 0$ for $u > 0$ and $\int_{0+} \frac{u^{p-1}}{\kappa^p(u)} du = +\infty$ such that $dP \times dt - a.e.$,

$$\forall y_1, y_2 \in \mathbf{R}^k, z \in \mathbf{R}^{k \times d}, \quad |g(\omega, t, y_1, z) - g(\omega, t, y_2, z)| \leq \kappa(|y_1 - y_2|).$$

In the following, we show (H1*) \implies (H1). In fact, if g satisfies (H1*), then we have, $dP \times dt - a.e.$,

$$\forall y_1, y_2 \in \mathbf{R}^k, z \in \mathbf{R}^{k \times d}, \quad |g(\omega, t, y_1, z) - g(\omega, t, y_2, z)|^p \leq \rho(|y_1 - y_2|^p),$$

where $\rho(u) = \kappa^p(u^{\frac{1}{p}})$. And, we have also,

$$\int_{0+} \frac{du}{\rho(u)} = \int_{0+} \frac{du}{\kappa^p(u^{\frac{1}{p}})} = \int_{0+} \frac{pu^{p-1}}{\kappa^p(u)} du = +\infty. \quad (20)$$

Thus, in view of Remark 2, we know that (H1) is true. Therefore, from Theorem 1 the following corollary is immediate. And, by Hölder's inequality we know that it generalizes the corresponding result in [2], where $p = 2$ and (H3) is replaced by $g(\cdot, 0, 0) \in M^2(0, T; \mathbf{R}^k)$:

Corollary 3 Let g satisfy (H1*), (H2) and (H3). Then for each $\xi \in L^p(\mathbf{R}^k)$, the BSDE with parameters (ξ, T, g) has a unique solution in L^p .

Remark 3 According to the classical theory of uniformly continuous functions, we can assume that the $\kappa(\cdot)$ in (H1*) is a concave function. Thus, applying (18) in

Lemma 2, by letting $\rho(u) = \kappa^q(u^{1/q})$ and $r = p/q$ with $q > p$ we deduce that if $\int_{0+} \frac{du}{\kappa^q(u^{1/q})} = +\infty$, then $\int_{0+} \frac{du}{\kappa^p(u^{1/p})} = +\infty$. Thus, noticing (20) we know that the bigger the p , the stronger the (H1*).

To the end, we give an example.

Example 1 Let $g(t, y, z) = h(|y|) + |z| + |B_t|$, where

$$h(x) := x|\ln x|^{1/p} \cdot 1_{0 < x \leq \delta} + (h'(\delta-)(x - \delta) + h(\delta)) \cdot 1_{x > \delta}$$

with $\delta > 0$ small enough. It is clear that g satisfies (H2) and (H3). We can also prove that g satisfies (H1*) with $\kappa(\cdot) = h(\cdot)$ by verifying that $\int_{0+} \frac{u^{p-1}}{h^p(u)} du = +\infty$, $h(\cdot)$ is a sub-additive function and then the following inequality holds:

$$\forall y_1, y_2 \in \mathbf{R}^k, z \in \mathbf{R}^{k \times d}, \quad |g(\omega, t, y_1, z) - g(\omega, t, y_2, z)| \leq h(|y_1 - y_2|).$$

Thus, this generator g satisfies all conditions in Corollary 3. Consequently, the BSDE with parameters (ξ, T, g) has a unique solution in L^p for each $\xi \in L^p(\mathbf{R}^k)$.

Finally, it is worth mentioning that we can directly verify that for each $q > p$, $\int_{0+} \frac{u^{q-1}}{h^q(u)} du < +\infty$.

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